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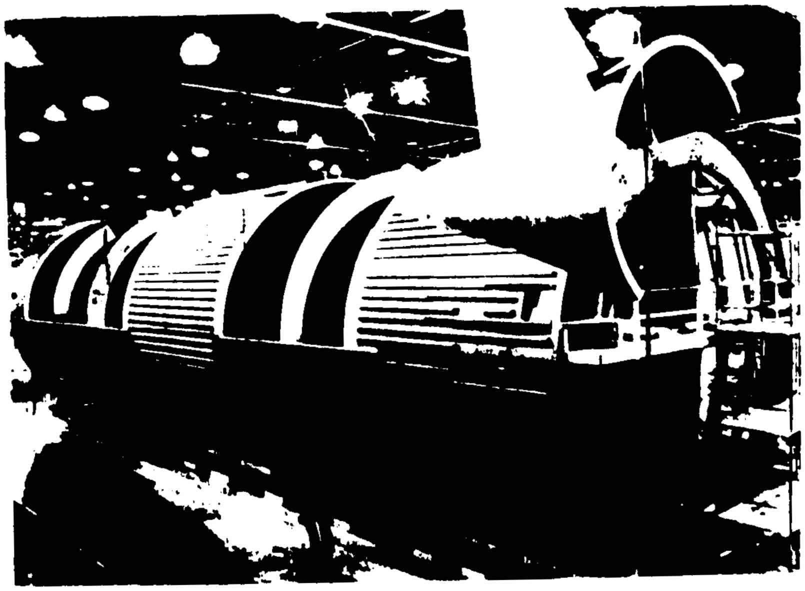
"Nell," a liquid-fueled rocket developed by Goddard (second from right), before launch on April 19, 1932 in New Mexico. A rocket of this type reached an altitude of 2 miles and a speed of 500 mph. Goddard developed, built and launched the world's first liquid-fueled rocket in 1926, six years before the earliest German work. At right, the successor to "Nell," 40 stories high when completed, and designed for the manned landing on the moon, in assembly at Michoud, La.



INTRODUCTORY LECTURES ON SPACE SCIENCE

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Chapter 1. THE LAWS OF MOTION

The study of motions and the forces which produce them is the essence of our attempt to understand our physical environment. The relation between applied forces and the resultant motions, which controls the working of the physical world, is contained in Newton's laws of motion, which were first published by him in 1686 in the Philosophiae Naturalis Principia Mathematica. Newton's formulation of these laws was so clear, and the application to celestial bodies so powerful and comprehensive, that his work appeared to illuminate a scene where all had been darkness before. Some years after Newton's death, Alexander Pope wrote,

> "Nature and nature's laws lay hid in night;
> God said, 'Let Newton be,' and all was light."

Pope's couplet underscored what was to become the traditional view, that Newton's work on the laws of motion and gravity stood by itself as the beginning of the scientific revolution. But Newton built on a foundation established by 2000 years of intelligent inquiry.

Aristotle dominated Western scientific thought for 2000 years prior to Newton. In the fourth century B.C. Aristotle abstracted certain observed properties of moving objects into a general law which survived for many centuries. Noting that an object propelled along the ground always comes to rest, he concluded that all motion is opposed by a natural resistance, which must be overcome by a propelling force. He proposed, therefore, as a general law of motion, that an object travels at a speed proportional to the force propelling it, and inversely proportional to the force resisting its motion. If F is the applied force, R the force of resistance, and v the resultant speed, Aristotle's law of motion may be written symbolically as

$$F = C \cdot \frac{v}{R}$$

in which C is a constant which does not depend on F, v, or R.

The difficulty with this law is that it predicts $v = 0$ when $F = 0$, i.e., the object comes to rest when the propelling force is removed. But in many cases that is not true. What force keeps an arrow moving after it leaves the bow, or a stone after it leaves the hand of the thrower?

Aristotle was in trouble because his law of motion required the continuous application of a force to sustain the motion. But he had an answer: air flows around the arrow or the stone from the front to the rear and pushes it ahead.

This answer is not very much to the taste of a modern mind, nor did it appeal to every scholar in the Middle Ages. In the sixth century A.D., Philoponus of Alexandria objected that the bow would not be needed if air alone could propel the arrow; that violent beating of the air behind the arrow would suffice. This was not observed; hence, Philoponus suggested that the action of the bow is essential; that the bow imparts a motive power to the arrow which persists for some time after the arrow is projected.

However, Philoponus thought that in empty space the motive power of a projectile would gradually disappear. He had not yet come upon the concept of inertia.

By the late Middle Ages scholarly activity had moved northward to include climates of severe winter cold, and ice skating on frozen lakes and rivers must have been a common experience. No doubt it was observed that the skater glides very far on an exceptionally smooth sheet of ice. Perhaps extrapolating from such simple observations,

a physicist named Jean Buriden, who taught at the University of Paris in the 14th century, drew the conclusion that an object which is set in motion acquires a property he called <u>impetus</u>, by which its motion is maintained indefinitely if no opposing force acts on it. He defined the impetus of a moving body as proportional to the product of its mass by its speed of motion. In these statements Buriden formulated elements of Newton's laws of motion three centuries before Newton's birth.

The essence of Aristotle's law of motion is that an object moves at a speed which is proportional to the force applied to it. Buriden broke this proportionality when he proposed that an object may continue in motion even though no force is applied. Yet the motion of an object is clearly influenced by applied forces in some way. The critical point which Aristotle missed is that an applied force changes the speed of an object. "Change" is the key word. When a propelling force is applied, the speed increases; when the force is removed, the speed remains constant thereafter. If an opposing force is applied, the speed decreases.

In physics, the rate of change of speed of a moving object is called its acceleration. The accelerated motion which results from the steady application of a constant force is called uniform acceleration.

Uniformly accelerated motion was investigated in detail by Galileo (1564-1642), who took as his example the falling motion of a body under the constant downward force of gravity. His objective was to determine the manner in which the speed changes as an object falls. Initially he held the erroneous view, expressed in a letter written in 1604, that a falling body picks up speed in proportion to the <u>distance</u> fallen. After some years he corrected his error and defined uniform acceleration as a steady increase of speed in proportion to the <u>time</u> elapsed in falling. This definition carries an implication which Galileo put to the test of experiment. If, starting from rest, a falling object gains speed in proportion to time, the velocity v at the end of a time t may be written

$$v = at$$

where a is the constant of proportionality. The constant a is the magnitude of the acceleration, since it controls the rate at which the speed increases. Let us now calculate the distance fallen in the time t. The average speed during the time t is $\frac{1}{2}v$, and the distance fallen is

$$\begin{aligned} d &= \tfrac{1}{2}v \times t = \tfrac{1}{2}(at) \times t \\ &= \tfrac{1}{2}at^2 \end{aligned}$$

Thus in uniform acceleration the distance fallen increases as the square of the time. To test this relation, Galileo constructed the following experiment. He cut a groove in a board 18 feet long, lined the groove with smooth parchment, and raised one end about two feet above the other. He permitted a small bronze ball, about an inch in diameter, to roll down the groove, and noted the time required for its descent. He then noted the time required for the ball to roll one quarter the distance previously gone, and found this time to be one half the previously measured time. Next he tried other distances of descent. In this way he determined that the distances traversed were in proportion to the squares of the times elapsed.

Accuracy was essential in these measurements, in order to distinguish the proposed laws of acceleration from other possible definitions, but accurate clocks were not available at that time for measurement of such short intervals of time as a few seconds. Galileo therefore devised a new instrument, a modification of the water clock, for his purposes. His clock consisted of a vessel of large horizontal dimensions, with a hole at the bottom which was covered by the finger until the ball was released at the top of the plane. When the ball reached one of the critical marks on the plane, the finger was replaced. In the interim water spurted from the hole into a pan which could be placed on a balance and weighed. The weight of water collected gave a measure of the time interval.

With the hindsight of 400 years Galileo's investigations seem very simple, but in his time they represented a highly original and difficult step. Galileo made a vital contribution to the laws of motion by shifting the emphasis from speed to acceleration, in the study of moving bodies, and by establishing the quantitative law of motion for the special case of acceleration under a constant force. But Galileo stopped short of attempting a general description of the motion produced by any kind of force. The general law was formulated by Isaac Newton, who was born on the manor of Woolsthorpe in England on Christmas Day of 1642, in the year of Galileo's death.

By Newton's time all the elements required for a complete understanding of the laws of motion were at hand. Several of his contemporaries arrived at some of his important results independently, in some instances preceding him, but Newton alone displayed the powers of analysis and intuition required to fit the pieces together. He started with a clear statement of the law of inertia:

> Every body continues in its state of rest or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed.

This was Newton's first law of motion. The second law drew on Galileo's work and on some parallel ideas of Descartes (1596-1650) on the relation between force and acceleration:

> An object is accelerated in proportion to the forces applied.

The force required to accelerate an object depends on the quantity of matter which the object contains; the larger the object, the greater the force which must be applied. Newton proposed that the force needed to produce a given degree of acceleration is exactly proportional to the quantity of matter in the object. For the quantity of matter he introduced the term "mass." If m is the mass of an object, a its acceleration, and F the force applied, the complete statement of the second law is

The second law provided the foundation for all developments in physics from the 17th through the 19th centuries; it lasted without modification for 219 years, from the publication of Newton's Principia in 1686 to the publication of Einstein's law of special relativity in 1905. Even today, it serves as an accurate description of all events which take place at speeds considerably less than the speed of light.

Note that the first law is included in the second, and need not be written separately; for, if the force applied to an object is zero, then the acceleration vanishes and the speed is therefore constant. Why did Newton write the same statement twice? Perhaps the burden of 2000 years of misconceptions regarding moving bodies weighed so heavily on his mind that he felt compelled to exhibit separately the aspect of his general law of motion which most clearly conflicted with older ideas.

The second law of motion sets acceleration proportional to applied force, whereas Aristotle's law had set speed proportional to force. It is impossible to overemphasize the importance of this simple change. In the study of moving objects it took us 2000 years to rid ourselves of the erroneous emphasis on speed rather than on acceleration.

The formulation of the laws of motion is completed by the third law:

> To every action there is opposed an equal reaction.

i.e., the action of one object on a second equals the reaction of the second back on the first.

When the objects are in contact, the third law is an obvious statement of everyday experience. A swimmer dives off a small boat into the water; he pushes back against the boat to get the impetus for his dive, and the boat pushes forward on him. As a consequence of the force which each exerts on the other, both move away from the point of contact in oopsite directions. The third law asserts that the force of the man on the boat is precisely equal to the reaction of the boat on the man.

Consider another example which brings out the importance of the relative masses of the two interacting objects. A man lies on the ground with his feet against a rock, and prepares to push the rock away by straightening his legs. If the rock is much larger than he is, by pushing against it he will only propel himself backward when he straightens his legs. The rock will move very slowly, if at all, in the other direction. On the other hand, if the rock is much smaller and lighter than the man, he will, as he straightens his legs in pushing against the rock, propel it away from him while he moves backwards slightly, if at all.

This thought-experiment suggests that when two objects separate by pushing on one another, they move apart at speeds which depend on their relative sizes or masses. More precisely, they separate at speeds which are inversely proportional to the ratio of their masses, as can be shown from the second law of motion. Labeling the objects by the numbers 1 and 2,

$$\frac{V_1}{V_2} = \frac{m_2}{m_1}$$

or, clearing fractions,

$$m_1 V_1 = m_2 V_2$$

The product of the mass and the velocity of a moving object is called its momentum. (It is the same quantity which Buriden suggested as a measure of impetus in the 14th century.) Thus, the third law states that when two objects, initially at rest, separate as a consequence of forces exerted on one another, they carry off equal amounts of momentum in opposite directions. This remark underlies the physics of rockets.

Chapter 2. THE MOTION OF THE MOON

It is a remarkable achievement on the part of Newton that he was able to distill all past experience regarding forces and motions into a simple and yet complete statement which provided the foundation of modern science, and stood unchanged for 200 years. Yet great as this achievement is, it does not compare with the quantitative formulation of the law of universal gravitation and the development of its consequences into a comprehensive system of the universe.

Several of Newton's predecessors and contemporaries had suggested that the gravity which pulls objects to the earth's surface is a universal force of attraction, which varies inversely as the square of the distance between the attracting bodies. Alphonse Borelli (1608-1678) anticipated Newton's *Principia* by suggesting, in 1666, that the attractive force of gravity is exerted by the sun on the planets, and bends their paths into ellipses. Robert Hooke (1635-1703) proposed as early as 1665, in the *Micrographia*, that gravity acts on the moon, the planets and all celestial bodies; his reason being that a force of attraction to the center must be introduced to explain the spherical shape of these objects.

named Johannes Kepler (1571-1630). Kepler had analyzed the observations on the motions of the planets, and from them he had derived three general laws of planetary motion:

1. The orbit of each planet is an ellipse with the sun at one focus.

2. The radius vector sweeps over equal areas in equal intervals of time.

3. The ratio of the cubes of the semimajor axes of the orbits of any two planets is equal to the ratio of the squares of their periods.

The first two of these were published in 1609, the same year in which Galileo completed his analysis of uniform acceleration. The third law appeared in 1618. Kepler arrived at these laws because he had available to him the superb measurements of Tycho Brahe on the positions of the planets. Kepler joined Brahe as his assistant in the observatory near Prague in 1600, and was assigned the orbit of Mars as his first task. Kepler tried to explain the observed positions of Mars in the sky by assuming that the planet traversed a circular orbit around the sun, following the traditional point of view that the circle was the natural path for celestial bodies. He was forced to place the sun in an off-center position in order to improve the agreement between his theory and the observations. With the aid of this assumption, and after four years of calculations, he reduced the discrepancy between theory and observation to eight minutes of arc. Eight minutes of arc is the angle between the eyes of a person at a distance of one city block. Yet this minute angle signaled the fall of the Ptolemaic epicycle and the rise of the new science.

In Ptolemy's description of the universe the sun, moon and planets moved in circular orbits about the earth, which stood at the center of the solar system. In order to account for the complicated motions of the other planets as seen from the earth, Ptolemy (2nd century A.D.) constructed a system of epicycles for their orbits. These were small circular motions superimposed on the main orbit of the planet around the earth. Copernicus (1473-1543) simplified the Ptolemaic system by placing the sun at the center of the solar system, but his description of planetary motions still relied on the epicycle. Kepler was the first astronomer to overthrow the circle and the epicycle, and establish the ellipse as the correct description of the motion of the planets. In so doing, he opened the way to Newton's subsequent linking of the elliptical orbit and the inverse square law of gravity.

The accuracy of Tycho Brahe's observations played a critical role in this development. Ptolemy and Copernicus based their epicycle theories on observations of the positions of the planets which were accurate to ten minutes of arc, and were able to fit the epicycle to the observed motions of the planets if they allowed this amount of error. The discrepancy of eight minutes of arc which Kepler found in his theory of Mars would have satisfied Ptolemy and Copernicus. But Brahe claimed his observations to be accurate to two minutes of arc, and Kepler believed him. He continued his labors through two more years and 900 folio pages of 16th century algebra and geometry, until he discovered that if Mars moved, not in a circle, but in an <u>ellipse</u> with the sun at one focus, then the positions of the planet could be predicted with an error of less than two minutes of arc at each point.

It was a fortunate circumstance that Brahe assigned Mars to Kepler on his arrival in Prague, because the orbit of Mars is more elliptical, that is, more egg-shaped, than the orbit of any other nearby planet. The earth's orbit is close to a perfect circle; it departs from circularity by two per cent. The orbit of the planet Venus departs from a circle by only 0.7 per cent. The orbit of Mars, however, is rather different; it departs from circularity by 9 per cent. Because the departure was as great as 9 per cent Kepler was unable to fit his calculations of the Mars orbit into the conventional ideas of circular orbits.

Kepler's second law does not concern us here, except to say that it tells us a planet moves fastest when it is closest to the sun and slowest when it is at its furthest distance in its elliptical orbit.

Kepler spent 22 years in extracting the third law of planetary motion from the data. The third law deals with the relation between the distance from a planet to the sun and the time it takes that planet to go around the sun, that is, with the length of its year. It may be stated as follows: The square of the time (T) in which a planet completes one circuit around the sun is proportional (proportionality symbol: $\propto$) to the cube of its distance to the sun (R) :

$$T^2 \propto R^3$$

As illustrations of the third law consider the orbits of the earth and Mars. The earth is 93 million miles from the sun, and revolves about the sun in 365 days or approximately 30 million seconds. Mars is 142 million miles from the sun, or roughly 1.5 times farther than the earth. Since $T^2 \propto R^3$ we have

$$\frac{T^2\,(\text{Mars})}{T^2\,(\text{Earth})} = \frac{R^3\,(\text{Mars})}{R^3\,(\text{Earth})} = 1.5^3 = 3.38$$

and

$$\frac{T\ (\text{Mars})}{T\ (\text{Earth})} = \sqrt{3.38} = 1.9$$

hence the Martian year is 1.9 times the length of a year on the earth, or approximately 690 days.

From the age of 25 to 47 Kepler labored over Brahe's data, testing them for significant relationships. He succeeded totally; within the observational accuracy of 2 minutes of arc there are no other laws to be found in the movement of the planets. Kepler found the strength needed to carry him through this ordeal in his belief that a divine order and purpose must exist in nature. He saw their expression in the beautiful simplicity of the proportion between the square of the time of revolution of a planet and the cube of its distance from the sun.

Newton also believed that the mechanics of the solar system expressed a divine purpose. Yet he could not be satisfied with Kepler's laws as they stood. Newton had developed a clear understanding of the laws of motion; he knew that a planet or any object moves in a straight line unless a force deflects it; he knew, therefore, that a force must act on the planets to keep them circling around the sun; he knew that a force must act on the moon to keep it circling around the earth; and he looked to Kepler's laws to provide the key to the nature of that force.

<u>The Law of Gravity</u>. Newton claimed that he was a very young man when he first had the remarkable thought on the universality of gravity. He had returned home from Cambridge University in 1665, at the age of 23, to avoid the Great Plague which struck the city in 1665 and 1666. He remained at home for two years, and, according to his recollections in later years, it was during this period of isolation that he first reflected on the motion of the moon, which revolves around the earth in an approximately circular orbit as the planets revolve around the sun. The moon would fly off at a tangent, were there not a force attracting it to the earth and constraining it to move in a circular path. What is this force? Could it be the same as the force of gravity which the earth exerts on objects at its surface? Newton conjectured that the two forces were identical. If this proposal were correct it would unite the earth and the heavens in one mechanical system. The boldness of the step cannot be exaggerated.

Newton turned to the motions of the moon and planets for a test of his hypothesis. Kepler had already written in 1609, "... the attractive force of the earth ... extends to the moon and even farther." Hooke had made the same conjecture. But Newton went much further. From Kepler's laws of planetary motion he derived a basic property of the force which holds the planets in their orbits. He then carried out calculations to determine whether this same property was possessed by the force that holds the moon in its orbit and the force that draws bodies to the ground. He writes, "And the same year (1665 or 1666) I began to think of gravity extending to the orb of the Moon, and having found out how to estimate the force with which a globe revolving within a sphere presses the surface of the sphere, from Kepler's Rule of the periodical times of the Planets being in a sesquialternate proportion of their distances from the centers of their Orbs I deduced that the forces which keep the Planets in their Orbs must (be) reciprocally as the squares of their distances from the centers about which they revolve: and thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth, and found them answer pretty nearly. All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for investigation, and minded Mathematicks and Philosophy more than at any time since. What Mr. Hugens has published since about centrifugal forces I suppose he had before me." That is, he combined the law of centrifugal reaction with Kepler's third law of planetary motion to derive the inverse square law for the variation of the gravitational force with distance from the attracting body (Chapter 3).

Armed with the $1/R^2$ law, Newton could now test the universality of gravity. He reasoned: if the force of gravity acts between all objects in the universe, then this force, which emanates from planets to the sun and binds these planets in their orbits, will also emanate from the earth, and from every other planet, and attract objects to them as well. This same attraction must be the familiar force of gravity which pulls objects to the surface of the earth; but the earth's gravity must, in that case, also reach out to the moon, and hold the moon in its orbit. Since the force of the sun's gravity falls off as $1/R^2$, the earth's gravity must also decrease in the same proportions. The moon is 240,000 miles from the earth, or 60 times farther removed from the center of the earth than a body on the surface of our planet. Therefore, the force of the earth's gravity, acting on the moon, should be weaker than the force of gravity acting at the surface of the earth in the ratio of 1 to $(60)^2$ or 1/3600.

Newton calculated the force required to keep the moon in its circular orbit around the earth, compared it to the force of gravity acting on objects at the surface, and found their ratio to be close to 1/3600. In his words, he found them to agree "pretty nearly."

Although these calculations are supposed to have been carried out by Newton in 1666, he did not publish or refer to them at that time, possibly because he had to make an assumption that he was unable to prove, namely, that the earth attracted an object at its surface with the same force that it would exert if all the mass of the planet were concentrated in a point at its center. That mathematical demonstration, if it could be discovered, would tie together theories and observations of all celestial motions in a single system. In 1679 he was able to prove this statement with the aid of the calculus.

In the interim other scientists had also derived the inverse square law of gravity by combining the law of centripetal acceleration with the third law of Kepler. Three men in England carried out the derivation in 1679. These were the physicist, Robert Hooke (1635-1703), the astronomer, Edmund Halley (1656-1742), and the architect, Christopher Wren (1632-1723). They were all members of the Royal Society, and all exceedingly interested in natural philosophy. When they had derived the inverse square law of gravity from Kepler's third law, the burning question for these men was whether Kepler's first law of elliptical orbits could be derived from the inverse square law for the gravitational force. If this were so, all known laws of planetary and terrestrial motion would be united in one system.

In the period of 1683-1684 the three men met several times to discuss the proposition. They guessed that it was true, but could not carry out the mathematics needed to complete the proof. Halley visited Newton later in 1684 to ask if he could help his friends in their problem. Newton replied that he had proved the result several years earlier, but was unable to find his old calculations. Under the stimulus of Halley's interest he returned to the problem he had laid aside years ago, and expanded his investigations into a major treatise, which appeared in 1687 under the imprimatur of the Royal Society with the title, Philosophiae Naturalis Principia Mathematica.

Chapter 3. EXERCISES IN GRAVITY

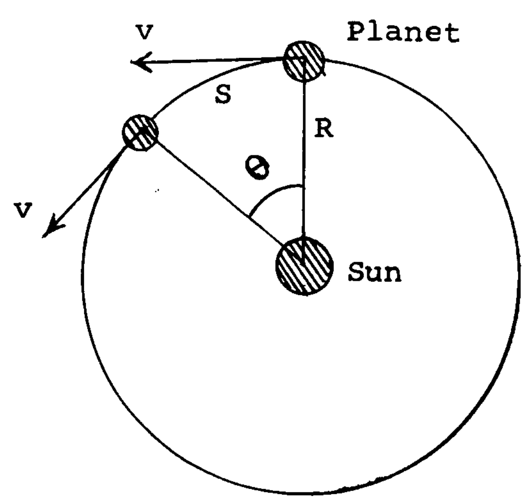
The derivation of the inverse square law of gravity from the third law of Kepler is among the most important calculations ever performed in the history of science. I include it, then, for the mathematically inclined reader who would like to retrace the steps of Newton's reasoning.

In the second law of motion

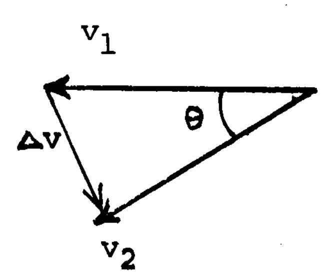
$$F = ma$$

F is the force acting on the planet, m is its mass, and a is its acceleration in the motion around the sun. The problem Newton set himself was to determine the force by deducing the acceleration from observations on the motion of the planet.

In Newton's first attack he probably replaced the elliptical orbits of the planets by circles in order to simplify his mathematical labors. Although the difference between the circle and the ellipse played a critical role in the subsequent history of the theory of gravity, it does not have a significant effect on the derivation of the inverse square law. Assuming, then, that the orbits of the planets are circles, the first task is to calculate the acceleration of a planet moving in a circle. In circular motion a planet keeps a constant distance from the sun, and we know, from Kepler's second law (Chapter 2, pp. 2-4), that in this case the speed of the planet in its orbit is also constant. If the speed is constant it seems that the acceleration must be zero, but it must be remembered that the motion of the planet is described not only by its speed but also by the direction of travel. Although the speed is constant the direction of motion changes continuously as the planet circles about the sun. This change of direction is also an acceleration. The particular kind of acceleration associated with circular motion is called centripetal acceleration. Its magnitude depends on the radius of the circle and the speed with which the planet moves. In order to calculate it, consider the diagram: a planet is shown moving in a circular orbit of radius T , the motion of the planet being indicated by an arrow labelled v (for velocity). In a short interval of time t the planet progresses a distance S in its orbit, the distance being given by S = vt. In this same interval of time the line



from the planet to the sun turns through an angle $\theta = S/R = vt/R$. The acceleration is the rate at which the velocity changes, i.e., the change in the velocity divided by the time interval t in which the change occurs. The next diagram shows the arrows representing the velocity at the beginning and end of the time interval, when the planet is at positions 1 and 2, respectively. The angle between the velocities v_1 and v_2 equals the angle θ between the lines joining the planet to the sun. The change of velocity in the time interval t is given by the length Δv of the line joining v_1 and v_2 . This is $2v \sin \frac{\theta}{2}$. If t is a short time, and, therefore, θ is a small angle, $\sin \frac{\theta}{2}$ is approximately equal to $\frac{\theta}{2}$ and



$$v = 2v \sin \frac{\theta}{2} = v\theta = \frac{v^2 t}{R}$$

The centripetal acceleration is

$$a = \text{velocity difference} \div \text{time interval}$$

$$= \frac{v^2 t}{R} \div t = \frac{v^2}{R}$$

This important formula apparently was derived by Newton in his youth in the period 1665-1667, according to his recollection 30 years later, but he did not communicate the result to anyone at the time. It was also derived by Christian Huyghens (1629-1695), a Dutch physicist, and published by him in 1673.

Knowing the formula for centripetal acceleration, we return to Newton's law of motion, and write

$$F = ma = m\frac{v^2}{R}$$

This formula gives the magnitude of the force F which must be applied to the planet to bend its path into a circular orbit of speed v and radius R. If v could be expressed in terms of the orbital distance R, we would have a formula for the gravitational force in terms of the distance between the planet and the sun, and our task would be completed. It is precisely this relation between v and R which is supplied by Kepler's third law of planetary motion. The third law states that

$$T^2 = C \cdot R^3$$

where T is the orbital period, or time for completion of an orbit, and R is defined above. C is a constant which has the same value for all planets. The speed of the planet is

$$v = \frac{\text{Distance around the orbit}}{\text{Time of one circuit}} = \frac{2\pi R}{T}$$

Thus,

$$F = m\frac{v^2}{R} = m\frac{4\pi^2 R^2}{R T^2}$$

and, eliminating T^2 with the aid of Kepler's third law,

$$F = m\frac{4\pi^2}{c}\frac{1}{R^2}$$

Thus the force holding a planet in its orbit falls off as the square of the distance (R) to the sun.

Once this result was in hand, Newton proceeded to test the universality of gravitation by determining whether the same inverse square law holds for the separate forces which bind the planets to the sun, hold the moon in its orbit, and pull objects to the surface of the earth. Again using the centripetal acceleration formula, he calculated the force holding the moon in its orbit around the earth. He then calculated the force with which the earth attracts an object lying on its surface. For this purpose he had to assume that the earth attracts bodies on its surface as if its matter were concentrated at the center. The distance with which the attraction acts is, therefore, the radius of the earth, or 4000 miles. The moon is 240,000 miles from the center of the earth. If the inverse square law is universally applicable, the force of gravity on the moon will be weaker than on objects at the earth's surface, in the ratio of $(4000/240{,}000)^2$ or one part in 3600.

In Newton's calculations he did not consider the actual forces exerted by the earth on the moon and on a falling object near its surface, respectively. Instead, he considered the accelerations produced by these forces, which are proportional to them according to his second law of motion. The ratio of the forces is, therefore, equal to the ratio of the accelerations. We will now calculate these accelerations as Newton did, and compare their ratio with 1:3600.

First we consider the gravitational acceleration of an object at the surface of the earth. This is readily obtained by experiments. A freely falling object experiences the full force of gravity, which causes it to accelerate in such a way that, as Galileo determined, the speed of falling increases in proportion to the time, and the distance fallen increases as the square of the time. If the falling object is initially at rest, and acquires a speed v at the end of a time t, then the acceleration is

$$\begin{aligned} a &= \text{change in speed} \div \text{time elapsed} \\ &= v/t \end{aligned}$$

The distance fallen is equal to the average speed during the fall, or $\frac{1}{2}v$, times t or $\frac{1}{2}vt$. In terms of acceleration,

$$d = \tfrac{1}{2}at^2$$

Therefore, a measurement of the distance d through which a body falls in a time t will yield the gravitational acceleration, a .

In Newton's time these measurements had been carried out on falling bodies, and it had been determined that in the first second of time an object falls 16 feet. Thus

$$16 \text{ feet} = \tfrac{1}{2}a(1 \text{ sec})^2$$

and

$$a = 32 \text{ feet/sec}^2$$

With the acceleration of a body at the surface of the earth determined, Newton then proceeded to calculate the acceleration of the moon toward the earth, using, once again, the Newton' Huyghens formula for centripetal acceleration:

$$a = v^2/R \ .$$

The moon traverses a circle of 240,000 mile radius and length 2   x 240,000 miles in a time of 28 days or (28 x 86,400) seconds, its orbital speed v being

$$v = \frac{2 \quad \times 240{,}000 \text{ miles}}{28 \times 86{,}400 \text{ seconds}} = 0.62 \text{ miles/second}$$

$$= 3260 \text{ feet/sec}$$

Thus the centripetal acceleration of the moon in its orbit is

$$a = \frac{(3260)^2}{(240{,}000 \times 5280)} \text{ feet/sec}^2$$

$$a = .0085 \text{ feet/sec}^2$$

The ratio of the moon's acceleration to the acceleration of gravity is

$$.00845 \div 32.2 = 1:3800$$

This result is close to the predicted ratio of 3600, the discrepancy being the result of approximations in our arithmetic. In Newton's calculations, when he repeated them in 1684 with the best values of a and R then available, the ratio of the accelerations agreed with the theory to 1/60 of one per cent, which was good enough to give strong support to the hypothesis of universal gravitation.

Newton assumed in his initial deliberations in 1665-67 that the force of the earth on a nearby object could be replaced by the attraction of a point mass situated at the center of the earth. This is an accurate assumption for the moon, which is so far away that it may be considered to be at very closely the same distance from every element of mass within the earth. But the force of gravity on an object at the surface of the earth is compounded of the forces exerted by all the elements of matter in the earth, some of which may be as close as a few feet, while others are as far away as the full diameter of the planet. All exert an attraction proportional to the inverse square of their individual distances from the object in question, and it is difficult to determine how these separate forces, whose magnitudes vary so greatly shall be added together. Newton only succeeded in solving this problem in 1679 or 1680. It is possible that he withheld the publication of his work on universal gravitation and the motion of the moon because he was unable to prove the validity of his assumption.

Complete Statement of the Law of Gravity. Newton stated that the gravitational attraction of the earth on the moon is inversely proportional to the square of the distance between them, and proportional to the product of the masses of both bodies; and, in general, that the gravitational force of attraction between two objects, of masses m and M, separated by a distance R, is

$$F = G \frac{mM}{R^2}$$

in which G is the constant of proportionality, called the constant of universal gravitation. In the metric system, with length, mass, and time measured in centimeters, grams and seconds, respectively, G has the value 6.7×10^{-8}. This is the force holding a planet in its orbit. If we combine the law of gravity with the second law of motion we derive the value of the orbital velocity of a planet at a distance R from the sun:

$$G \frac{mM}{R^2} = ma = M \frac{v^2}{R}$$

$$v = \sqrt{\frac{MG}{R}}$$